Further Mathematics

Subject	Qualification	Examination Board
Further Mathematics	A Level (or an AS level studied over two years).	Edexcel
Additional Information:	You must also be taking Mathematics A level.	

Task Overview:
To begin to explore the topic of matrices in mathematics. To become familiar with some of the technology you will use as part of your Further Mathematics course. To complete exercises on this topic and make use of relevant technology. To be able to experiment and discover some new mathematics.
Success Criteria:
Exercises successfully completed (evidence on paper ready to hand in, marked and corrected as far as possible). Notes made on any areas of difficulty that need further work or extra support. Ability to use the new Classwiz calculator. Have gained an understanding of matrices and matrix calculations.
Resources:
We will provide the exercises, instructions for the task and an outline of the investigation extension. Note that most of this can be completed without the technology. You will need to buy the Casio fx-991EX Classwiz calculator (this is a requirement for both the Mathematics and Further Mathematics courses). However, don't worry if you don't have this until later.
How will the work produced will fit into subsequent work and the specification as a whole?
The knowledge you gain of matrices will give you a head start in your studies in both the Core Pure Maths and Decision Maths parts of your Further Mathematics course. Familiarity with the new technology will enhance your studies throughout the course.
How should the work should be presented?
Evidence on A4 paper clearly labelled and communicated / explained as appropriate. Remember to mark all exercises in red as you go, stopping to correct in red where you have made mistakes. You should make a note of any problem areas and problems or successes with using the technology.

Who should you contact if you should require further assistance with the work before the end of term?	Mrs Wilshire. Teacher of Further Mathematics. s.wilshire@gildredgehouse.org.uk
Length of time expected to complete tasks:	3-5 hours.
Submission Requirements:	Work should be ready to hand in at your first lesson in Further Mathematics.

What equipment will be needed for the subject?
See above for calculator requirement. You will also need to obtain the following textbooks ready for the start of the course: Pearson Core Pure Mathematics Book 1/AS ISBN 978 – 1 – 292 – 18333-6. Pearson Decision Mathematics 1 D1 ISBN 978-1-292-18329-9. You will need the usual mathematics equipment you used for GCSE e.g. blue or black pen(s), pencil, ruler, rubber, protractor, compass, red pen for marking, and whiteboard pen(s) for use on small whiteboards. In addition for A Level you need a supply of A4 lined paper and a filing system e.g. A4 folder and dividers. There are two further textbooks required later for year 13.
Optional Extension Task/Further Reading
Extend your work into matrix transformations in 2 and possibly 3 dimensions and / or research how matrices are used in mathematics and related areas. There are applications in Computer Science, Business and Engineering.

Transition Task: Further Mathematics

You will be learning about the topic of Matrices.

The skills you learn here will be used in both the Core Pure and Decision Mathematics areas of study in your A Level Further Mathematics course.

There are three sections to this booklet

1. Matrices
2. Multiplying Matrices
3. Transformations.

You are required to complete the first two sections. Section 3 is an optional extension to this work.

For sections 1 and 2:

Read through the information given, stopping at each discussion point, technology box and activity. Here you should answer the questions or find out how to use the technology (your calculator).

There are answers at the back. Please mark and correct your work in red to make sure you have understood before moving on.

Then complete the exercise. You should answer all questions in Exercise 1.1 and questions 1 to 9 of Exercise 1.2. (Show any working clearly). Where a question has multiple very similar parts you can just do the first and last part, mark your work, and if it's all correct and you fully understand move on to the next question. If it's not all correct, or you need more practice, do all parts of the question.

If you are feeling confident you could try Exercise 1.2 questions 10 and 11 which are more challenging.

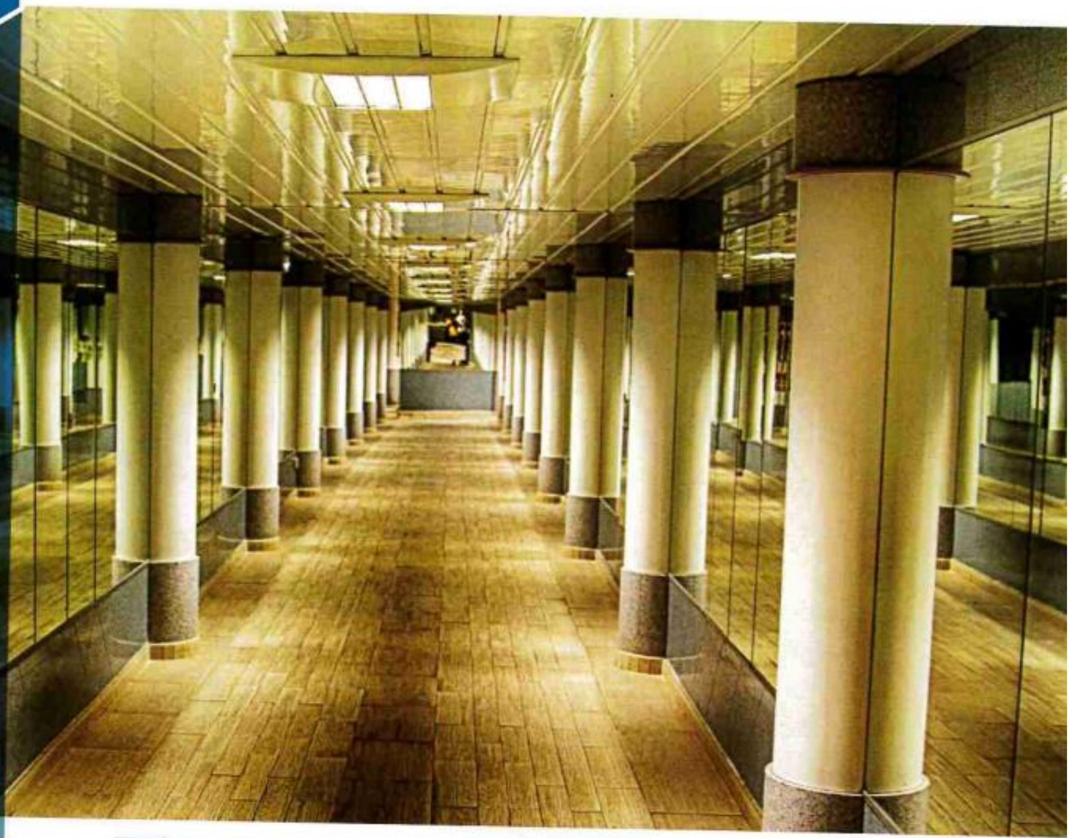
Use the answers at the back. Be sure to mark and correct your work in red to make sure you have understood before moving on.

Optional extension / investigation: Section 3:

If you would like to you can investigate how matrices can be used to transform shapes. Or you might want to explore how matrices are used in business, engineering or computer science.

Please hand in all work on A4 paper clearly named and labelled with topic and section titles. Work should be handed in at your first Further Mathematics lesson.

1 Matrices and transformations



As for everything else, so for a mathematical theory – beauty can be perceived but not explained.

Arthur Cayley 1883

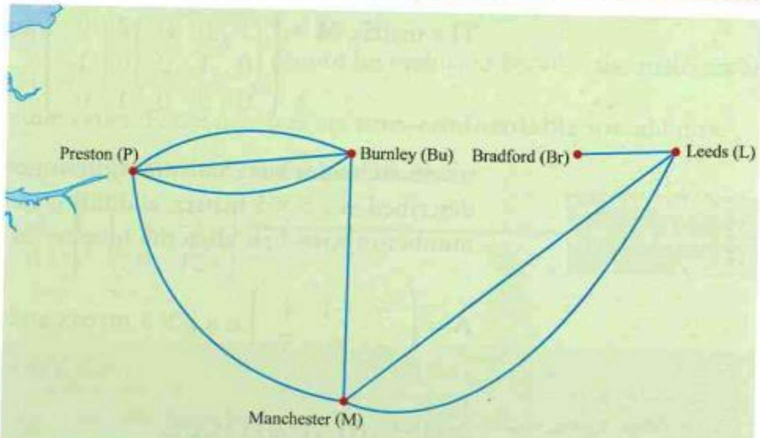


Figure 1.1 Illustration of some major roads and motorways joining some towns and cities in the north of England.

Discussion point

→ How many direct routes (without going through any other town) are there from Preston to Burnley? What about Manchester to Leeds? Preston to Manchester? Burnley to Leeds?

1 Matrices

You can represent the number of direct routes between each pair of towns (shown in Figure 1.1) in an array of numbers like this:

	Br	Bu	L	M	P
Br	0	0	1	0	0
Bu	0	0	0	1	3
L	1	0	0	2	0
M	0	1	2	0	1
P	0	3	0	1	0

This array is called a matrix (the plural is matrices) and is usually written inside curved brackets.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 1 & 0 \end{pmatrix}$$

It is usual to represent matrices by capital letters, often in bold print.

A matrix consists of rows and columns, and the entries in the various cells are known as **elements**.

The matrix $\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 1 & 0 \end{pmatrix}$ representing the routes between the towns and cities has 25 elements, arranged in five rows and five columns. **M** is described as a 5×5 matrix, and this is the **order** of the matrix. You state the number of rows first, then the number of columns. So, for example, the matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 0 & 5 \end{pmatrix}$ is a 2×3 matrix and $\mathbf{B} = \begin{pmatrix} 4 & -4 \\ 3 & 4 \\ 0 & -2 \end{pmatrix}$ is a 3×2 matrix.

Special matrices

Some matrices are described by special names which relate to the number of rows and columns or the nature of the elements.

Matrices such as $\begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 5 & 1 \\ 2 & 0 & -4 \\ 1 & 7 & 3 \end{pmatrix}$ which have the same number of rows as columns are called **square matrices**.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 **identity matrix** or **unit matrix**, and similarly $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called the 3×3 identity matrix. Identity matrices must be square, and are usually denoted by **I**. An identity matrix consists of 1's on the leading diagonal (the diagonal from top left to bottom right) and 0's everywhere else.

The matrix $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is called the 2×2 **zero matrix**. Zero matrices can be of any order.

Two matrices are said to be **equal** if and only if they have the same order and each element in one matrix is equal to the corresponding element in the other matrix. So, for example, the matrices **A** and **D** below are equal, but **B** and **C** are not equal to any of the other matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Working with matrices

Matrices can be added or subtracted if they are of the same order.

$$\begin{pmatrix} 2 & 4 & 0 \\ -1 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 4 \\ 2 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 4 \\ 1 & 3 & 0 \end{pmatrix}$$

Add the elements in corresponding positions.

$$\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 7 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 5 & -1 \end{pmatrix}$$

Subtract the elements in corresponding positions.

But $\begin{pmatrix} 2 & 4 & 0 \\ -1 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$ cannot be evaluated because the matrices are not of the same order. These matrices are **non-conformable** for addition.

You can also multiply a matrix by a **scalar** number:

$$2\begin{pmatrix} 3 & -4 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -8 \\ 0 & 12 \end{pmatrix}$$

Multiply each of the elements by 2.

TECHNOLOGY

You can use a calculator to add and subtract matrices of the same order and to multiply a matrix by a number. For your calculator, find out:

- the method for inputting matrices
- how to add and subtract matrices
- how to multiply a matrix by a number for matrices of varying sizes.

Associativity and commutativity

When working with numbers the properties of **associativity** and **commutativity** are often used.

Associativity

Addition of numbers is **associative**.

$(3 + 5) + 8 = 3 + (5 + 8)$

When you add numbers, it does not matter how the numbers are grouped, the answer will be the same.

Discussion points

- Give examples to show that subtraction of numbers is not commutative or associative.
- Are matrix addition and matrix subtraction associative and/or commutative?

Commutativity

Addition of numbers is **commutative**.

$4 + 5 = 5 + 4$

When you add numbers, the order of the numbers can be reversed and the answer will still be the same.

Exercise 1.1

① Write down the order of these matrices.

(i) $\begin{pmatrix} 2 & 4 \\ 6 & 0 \\ -3 & 7 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 8 & 4 \\ -2 & -3 & 1 \\ 5 & 3 & -2 \end{pmatrix}$ (iii) $\begin{pmatrix} 7 & -3 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & -6 & 4 & 9 \\ 5 & 10 & 11 & -4 \end{pmatrix}$ (vi) $\begin{pmatrix} 8 & 5 \\ -2 & 0 \\ 3 & -9 \end{pmatrix}$

② For the matrices

$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 7 & -3 \\ 1 & 4 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 3 & 5 & -9 \\ 2 & 1 & 4 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 0 & -4 & 5 \\ 2 & 1 & 8 \end{pmatrix}$

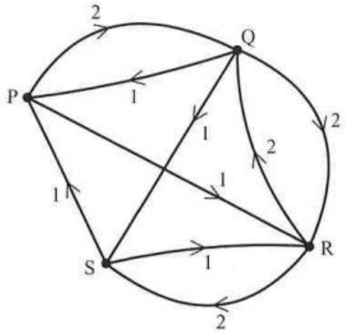
$\mathbf{E} = \begin{pmatrix} -3 & 5 \\ -2 & 7 \end{pmatrix}$ $\mathbf{F} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

find, where possible

(i) $\mathbf{A} - \mathbf{E}$ (ii) $\mathbf{C} + \mathbf{D}$ (iii) $\mathbf{E} + \mathbf{A} - \mathbf{B}$ (iv) $\mathbf{F} + \mathbf{D}$ (v) $\mathbf{D} - \mathbf{C}$
(vi) $4\mathbf{F}$ (vii) $3\mathbf{C} + 2\mathbf{D}$ (viii) $\mathbf{B} + 2\mathbf{F}$ (ix) $\mathbf{E} - (2\mathbf{B} - \mathbf{A})$

③ The diagram in Figure 1.2 shows the number of direct flights on one day offered by an airline between cities P, Q, R and S.

The same information is also given in the partly completed matrix **X**.



$$\mathbf{X} = \begin{matrix} & \text{To} \\ & \begin{matrix} P & Q & R & S \end{matrix} \end{matrix}$$

$$\text{From}\quad \mathbf{X} = \begin{matrix} P \\ Q \\ R \\ S \end{matrix} \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & & & \\ & & & \\ & & & \end{pmatrix}$$

Figure 1.2

(i) Copy and complete the matrix **X**.

A second airline also offers flights between these four cities. The following matrix represents the total number of direct flights offered by the two airlines.

$$\begin{pmatrix} 0 & 2 & 3 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}$$

(ii) Find the matrix **Y** representing the flights offered by the second airline.

(iii) Draw a diagram similar to the one in Figure 1.2, showing the flights offered by the second airline.

④ Find the values of w, x, y and z such that

$$\begin{pmatrix} 3 & w \\ -1 & 4 \end{pmatrix} + x\begin{pmatrix} 2 & -1 \\ y & z \end{pmatrix} = \begin{pmatrix} -9 & 8 \\ 11 & -8 \end{pmatrix}.$$

⑤ Find the possible values of p and q such that

$$\begin{pmatrix} p^2 & -3 \\ 2 & 9 \end{pmatrix} - \begin{pmatrix} 5p & -2 \\ -7 & q^2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 9 & 4 \end{pmatrix}.$$

⑥ Four local football teams took part in a competition in which they each played each other twice, once at home and once away. Figure 1.3 shows the results matrix after half of the games had been played.

	Win	Draw	Lose	Goals for	Goals against
City	2	1	0	6	3
Rangers	0	0	3	2	8
Town	2	0	1	4	3
United	1	1	1	5	3

Figure 1.3

(i) The results of the next three matches are as follows:

City 2	Rangers 0
Town 3	United 3
City 2	Town 4

Find the results matrix for these three matches and hence find the complete results matrix for all the matches so far.

(ii) Here is the complete results matrix for the whole competition.

$$\begin{pmatrix} 4 & 1 & 1 & 12 & 8 \\ 1 & 1 & 4 & 5 & 12 \\ 3 & 1 & 2 & 12 & 10 \\ 1 & 3 & 2 & 10 & 9 \end{pmatrix}$$

Find the results matrix for the last three matches (City vs United, Rangers vs Town and Rangers vs United) and deduce the result of each of these three matches.

⑦ A mail-order clothing company stocks a jacket in three different sizes and four different colours.

The matrix $\mathbf{P} = \begin{pmatrix} 17 & 8 & 10 & 15 \\ 6 & 12 & 19 & 3 \\ 24 & 10 & 11 & 6 \end{pmatrix}$ represents the number of jackets in stock at the start of one week.

The matrix $\mathbf{Q} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 1 & 3 & 4 & 6 \\ 5 & 0 & 2 & 3 \end{pmatrix}$ represents the number of orders for jackets received during the week.

(i) Find the matrix $\mathbf{P} - \mathbf{Q}$.

What does this matrix represent? What does the negative element in the matrix mean?

A delivery of jackets is received from the manufacturers during the week.

The matrix $\mathbf{R} = \begin{pmatrix} 5 & 10 & 10 & 5 \\ 10 & 10 & 5 & 15 \\ 0 & 0 & 5 & 5 \end{pmatrix}$ shows the number of jackets received.

(ii) Find the matrix which represents the number of jackets in stock at the end of the week after all the orders have been dispatched.

(iii) Assuming that this week is typical, find the matrix which represents sales of jackets over a six-week period. How realistic is this assumption?

2 Multiplication of matrices

When you multiply two matrices you do not just multiply corresponding terms. Instead you follow a slightly more complicated procedure. The following example will help you to understand the rationale for the way it is done.

There are four ways of scoring points in rugby: a try (five points), a conversion (two points), a penalty (three points) and a drop goal (three points). In a match Tonga scored three tries, one conversion, two penalties and one drop goal.

So their score was

$3 \times 5 + 1 \times 2 + 2 \times 3 + 1 \times 3 = 26$.

You can write this information using matrices. The tries, conversions, penalties and drop goals that Tonga scored are written as the 1×4 row matrix $(3 \;\; 1 \;\; 2 \;\; 1)$ and the points for the different methods of scoring as the 4×1 column matrix

$$\begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$$

These are combined to give the 1×1 matrix $(3 \times 5 + 1 \times 2 + 2 \times 3 + 1 \times 3) = (26)$.

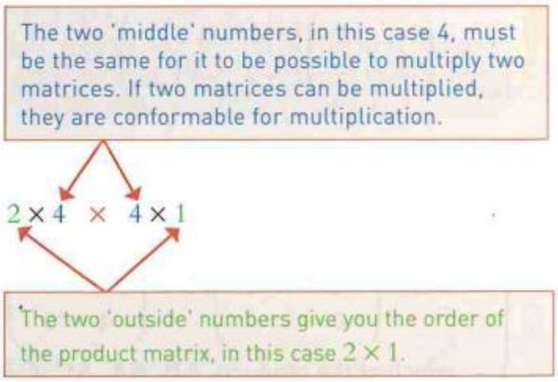
Combining matrices in this way is called **matrix multiplication** and this example is written as $(3 \;\; 1 \;\; 2 \;\; 1)\begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix} = (26)$.

The use of matrices can be extended to include the points scored by the other team, Japan. They scored two tries, two conversions, four penalties and one drop goal. This information can be written together with Tonga's scores as a 2×4 matrix, with one row for Tonga and the other for Japan. The multiplication is then written as

$$\begin{pmatrix} 3 & 1 & 2 & 1 \\ 2 & 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 26 \\ 29 \end{pmatrix}.$$

So Japan scored 29 points and won the match.

This example shows you two important points about matrix multiplication. Look at the orders of the matrices involved.



You can see from the previous example that multiplying matrices involves multiplying each element in a row of the left-hand matrix by each element in a column of the right-hand matrix and then adding these products.

Example 1.1

Find $\begin{pmatrix} 10 & 3 \\ -2 & 7 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Solution

The product will have order 2×1.

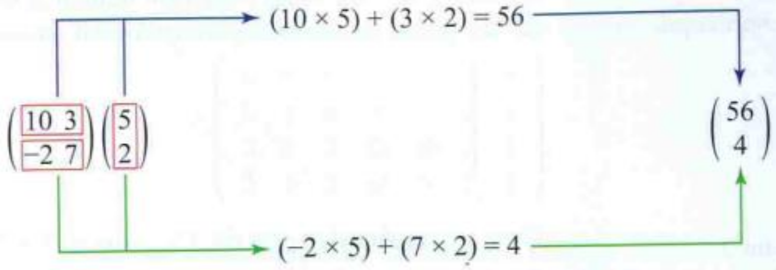


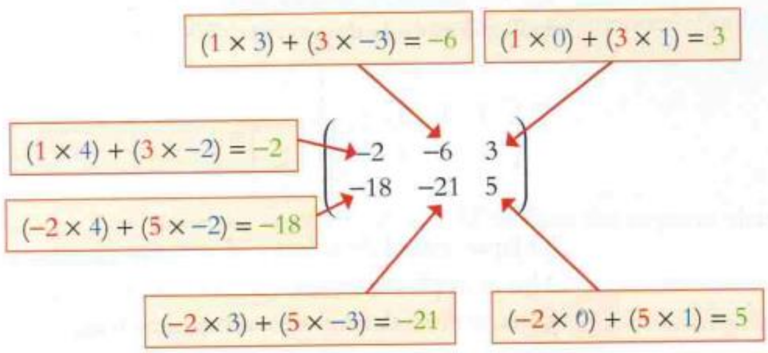
Figure 1.4

Example 1.2

Find $\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 0 \\ -2 & -3 & 1 \end{pmatrix}$.

Solution

The order of this product is 2×3.



So $\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 0 \\ -2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & 3 \\ -18 & -21 & 5 \end{pmatrix}$

Discussion point

➜ If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 4 & 1 \\ 0 & 3 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 8 & -1 \\ -2 & 3 \\ 4 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 & 0 \\ 3 & -4 \end{pmatrix}$

which of the products **AB**, **BA**, **AC**, **CA**, **BC** and **CB** exist?

Example 1.3

Find $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

What do you notice?

Solution

The order of this product is 2×2.

$$\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

$(3 \times 1) + (2 \times 0) = 3$

$(3 \times 0) + (2 \times 1) = 2$

$(-1 \times 0) + (4 \times 1) = 4$

$(-1 \times 1) + (4 \times 0) = -1$

Multiplying a matrix by the identity matrix has no effect.

Properties of matrix multiplication

In this section you will look at whether matrix multiplication is:

- commutative
- associative.

On page 4 you saw that for numbers, addition is both associative and commutative. Multiplication is also both associative and commutative. For example:

$$(3 \times 4) \times 5 = 3 \times (4 \times 5)$$

and

$$3 \times 4 = 4 \times 3$$

ACTIVITY 1.1

Using $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}$ find the products $\mathbf{AB}$ and $\mathbf{BA}$ and hence comment on whether or not matrix multiplication is commutative.

Find a different pair of matrices, $\mathbf{C}$ and $\mathbf{D}$, such that $\mathbf{CD} = \mathbf{DC}$.

TECHNOLOGY

You could use the matrix function on your calculator.

ACTIVITY 1.2

Using $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, find the matrix products:

(i) $\mathbf{AB}$
(ii) $\mathbf{BC}$
(iii) $(\mathbf{AB})\mathbf{C}$
(iv) $\mathbf{A}(\mathbf{BC})$

Does your answer suggest that matrix multiplication is associative?

Is this true for all 2×2 matrices? How can you prove your answer?

Exercise 1.2

In this exercise, do not use a calculator unless asked to. A calculator can be used for checking answers.

① Write down the orders of these matrices.

(i) (a) $\mathbf{A} = \begin{pmatrix} 3 & 4 & -1 \\ 0 & 2 & 3 \\ 1 & 5 & 0 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & 3 & 6 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 4 & 9 & 2 \\ 1 & -3 & 0 \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} 0 & 2 & 4 & 2 \\ 0 & -3 & -8 & 1 \end{pmatrix}$

(e) $\mathbf{E} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. (f) $\mathbf{F} = \begin{pmatrix} 2 & 5 & 0 & -4 & 1 \\ -3 & 9 & -3 & 2 & 2 \\ 1 & 0 & 0 & 10 & 4 \end{pmatrix}$

(ii) Which of the following matrix products can be found? For those that can state the order of the matrix product.

(a) $\mathbf{AE}$ (b) $\mathbf{AF}$ (c) $\mathbf{FA}$ (d) $\mathbf{CA}$ (e) $\mathbf{DC}$

② Calculate these products.

(i) $\begin{pmatrix} 3 & 0 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 4 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & -3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 5 & 8 \\ -3 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 5 & -1 & 0 \\ 3 & 6 & 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -9 \\ 11 \\ -2 \end{pmatrix}$

T Check your answers using the matrix function on a calculator if possible.

③ Using the matrices $\mathbf{A} = \begin{pmatrix} 5 & 9 \\ -2 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 5 \\ 2 & -9 \end{pmatrix}$, confirm that matrix multiplication is not commutative.

④ For the matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -3 & 7 \\ 2 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 1 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 3 & 4 \\ 7 & 0 \\ 1 & -2 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 4 & 7 \\ 3 & -2 \\ 1 & 5 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 3 & 7 & -5 \\ 2 & 6 & 0 \\ -1 & 4 & 8 \end{pmatrix}$$

calculate, where possible, the following:

(i) **AB** (ii) **BA** (iii) **CD** (iv) **DC** (v) **EF** (vi) **FE**

T ⑤ Using the matrix function on a calculator, find $\mathbf{M}^4$ for the matrix

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 4 & 3 \end{pmatrix}.$$

Note

$\mathbf{M}^4$ means $\mathbf{M} \times \mathbf{M} \times \mathbf{M} \times \mathbf{M}$

⑥ $\mathbf{A} = \begin{pmatrix} x & 3 \\ 0 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2x & 0 \\ 4 & -3 \end{pmatrix}$:

(i) Find the matrix product **AB** in terms of x.

(ii) If $\mathbf{AB} = \begin{pmatrix} 10x & -9 \\ -4 & 3 \end{pmatrix}$, find the possible values of x.

(iii) Find the possible matrix products **BA**.

⑦ (i) For the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, find

[a] $\mathbf{A}^2$

[b] $\mathbf{A}^3$

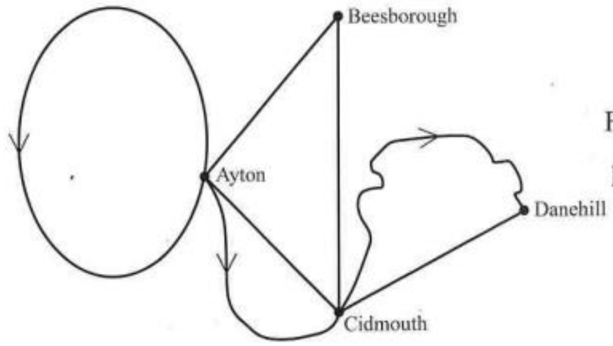
[c] $\mathbf{A}^4$

(ii) Suggest a general form for the matrix $\mathbf{A}^n$ in terms of n.

T (iii) Verify your answer by finding $\mathbf{A}^{10}$ on your calculator and confirming it gives the same answer as using (iv).

⑧ The map in Figure 1.5 below shows the bus routes in a holiday area. Lines represent routes that run each way between the resorts. Arrows indicated one-way scenic routes.

M is the partly completed 4 × 4 matrix which shows the number of direct routes between the various resorts.



		To			
		A	B	C	D
From	A	1	1	2	0
M =	B				
	C				
	D				

Figure 1.5

(i) Copy and complete the matrix **M**.

(ii) Calculate $\mathbf{M}^2$ and explain what information it contains.

(iii) What information would $\mathbf{M}^3$ contain?

⑨ $\mathbf{A} = \begin{pmatrix} 4 & x & 0 \\ 2 & -3 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 2 & -5 \\ 4 & x \\ x & 7 \end{pmatrix}$:

(i) Find the product **AB** in terms of x.

A symmetric matrix is one in which the entries are symmetrical about the leading diagonal, for example $\begin{pmatrix} 2 & 5 \\ 5 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 & -6 \\ 4 & 2 & 5 \\ -6 & 5 & 1 \end{pmatrix}$.

(ii) Given that the matrix **AB** is symmetric, find the possible values of x.

(iii) Write down the possible matrices **AB**.

⑩ The matrix **A**, in Figure 1.6, shows the number of sales of five flavours of ice cream: Vanilla(V), Strawberry(S), Chocolate(C), Toffee(T) and Banana(B), from an ice cream shop on each of Wednesday(W), Thursday(Th), Friday(F) and Saturday(Sa) during one week.

$$\mathbf{A} = \begin{matrix} & V & S & C & T & B \\ W & 63 & 49 & 55 & 44 & 18 \\ Th & 58 & 52 & 66 & 29 & 26 \\ F & 77 & 41 & 81 & 39 & 25 \\ Sa & 101 & 57 & 68 & 63 & 45 \end{matrix}$$

Figure 1.6

(i) Find a matrix **D** such that the product **DA** shows the total number of sales of each flavour of ice cream during the four-day period and find the product **DA**.

(ii) Find a matrix **F** such that the product **AF** gives the total number of ice cream sales each day during the four-day period and find the product **AF**.

The Vanilla and Banana ice creams are served with strawberry sauce; the other three ice creams are served with chocolate sprinkles.

(iii) Find two matrices, **S** and **C**, such that the product **DAS** gives the total number of servings of strawberry sauce needed and the product **DAC** gives the total number of servings of sprinkles needed during the four-day period. Find the matrices **DAS** and **DAC**.

The price of Vanilla and Strawberry ice creams is 95p, Chocolate ice creams cost £1.05 and Toffee and Banana ice creams cost £1.15 each.

(iv) Using only matrix multiplication, find a way of calculating the total cost of all of the ice creams sold during the four-day period.

⑪ Figure 1.7 shows the start of the plaiting process for producing a leather bracelet from three leather strands a, b and c.

The process has only two steps, repeated alternately:

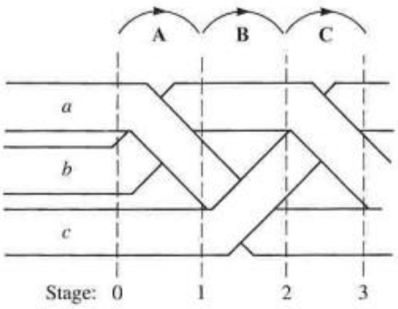


Figure 1.7

Step 1: cross the top strand over the middle strand

Step 2: cross the middle strand under the bottom strand.

At the start of the plaiting process, Stage 0, the order of the strands is given by $\mathbf{S}_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

(i) Show that pre-multiplying $\mathbf{S}_0$ by the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ gives $\mathbf{S}_1$, the matrix which represents the order of the strands at Stage 1.

(ii) Find the 3×3 matrix $\mathbf{B}$ which represents the transition from Stage 1 to Stage 2.

(iii) Find matrix $\mathbf{M} = \mathbf{BA}$ and show that $\mathbf{MS}_0$ gives $\mathbf{S}_2$, the matrix which represents the order of the strands at Stage 2.

(iv) Find $\mathbf{M}^2$ and hence find the order of the strands at Stage 4.

(v) Calculate $\mathbf{M}^3$. What does this tell you?

3 Transformations

You are already familiar with several different types of transformation, including reflections, rotations and enlargements.

- The original point, or shape, is called the **object**.
- The new point, or shape, after the transformation, is called the **image**.
- A transformation is a **mapping** of an object onto its image.

Some examples of transformations are illustrated in Figures 1.8 to 1.10 (note that the vertices of the image are denoted by the same letters with a dash, e.g. A′, B′).

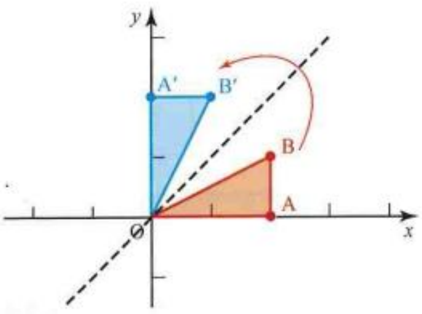


Figure 1.8 Reflection in the line $y = x$

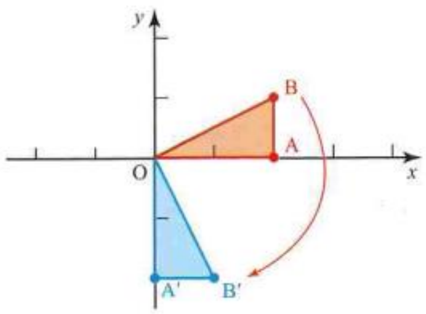


Figure 1.9 Rotation through 90° clockwise, centre O

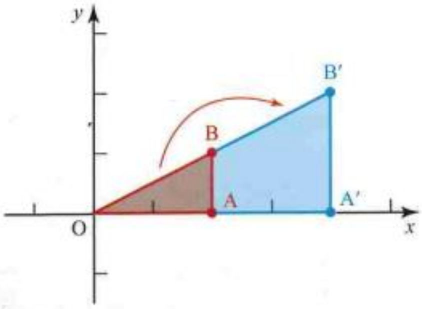


Figure 1.10 Enlargement centre O, scale factor 2

In this section, you will also meet the idea of

- a **stretch** parallel to the x-axis or y-axis

and three-dimensional transformations where

- a shape is reflected in the planes $x = 0$, $y = 0$ or $z = 0$
- a shape is rotated about one of the three coordinate axes.

A transformation maps an object according to a rule and can be represented by a matrix (see next section). The effect of a transformation on an object can be found by looking at the effect it has on the **position vector** of the point $\begin{pmatrix} x \\ y \end{pmatrix}$, i.e. the vector from the origin to the point (x, y). So, for example, to find the effect of a transformation on the point $(2, 3)$ you would look at the effect that the transformation matrix has on the position vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Vectors that have length or **magnitude** of 1 are called **unit vectors**.

In two dimensions, two unit vectors that are of particular interest are

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the x-axis

$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ – a unit vector in the direction of the y-axis.

The equivalent unit vectors in three dimensions are

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the x-axis

$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ – a unit vector in the direction of the y-axis

$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ – a unit vector in the direction of the z-axis.

Finding the transformation represented by a given matrix

Start by looking at the effect of multiplying the unit vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under this transformation is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

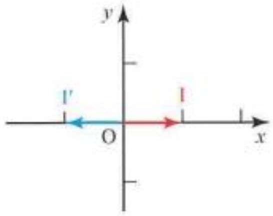


Figure 1.11

Note

The letter I is often used for the point (1, 0).

The image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under the transformation is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

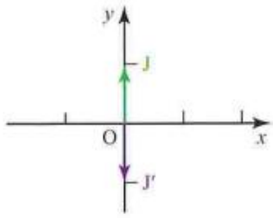


Figure 1.12

Note

The letter J is often used for the point (0, 1).

You can see from this that the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a rotation, centre the origin, through 180°.

Example 1.4

Describe the transformations represented by the following matrices.

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Solution

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

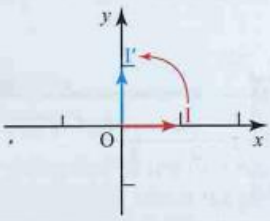


Figure 1.13

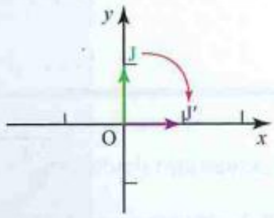


Figure 1.14

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.

(ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

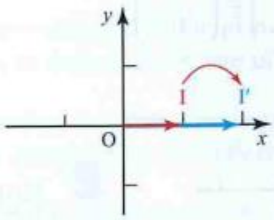


Figure 1.15

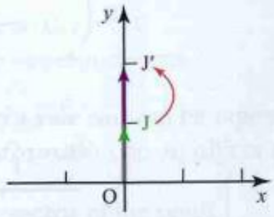


Figure 1.16

The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement, centre the origin, scale factor 2.

You can see that the images of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the two columns of the transformation matrix.

Finding the matrix that represents a given transformation

Hint

You may find it easier to see what the transformation is when you use a shape, like the unit square, rather than points or lines.

The connection between the images of the unit vectors i and j and the matrix representing the transformation provides a quick method for finding the matrix representing a transformation.

It is common to use the unit square with coordinates O(0, 0), I(1, 0), P(1, 1) and J(0, 1).

You can think about the images of the points I and J, and from this you can write down the images of the unit vectors **i** and **j**.

This is done in the next example.

Example 1.5

By drawing a diagram to show the image of the unit square, find the matrices which represent each of the following transformations:

(i) a reflection in the x-axis

(ii) an enlargement of scale factor 3, centre the origin.

Solution

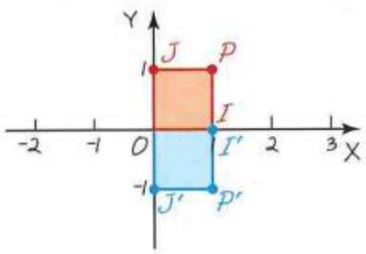


Figure 1.17

(i) You can see from Figure 1.17 that I (1, 0) is mapped to itself and J (0, 1) is mapped to J′ (0, −1).

So the image of $\mathbf{I}$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and the image of $\mathbf{J}$ is $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

So the matrix which represents a reflection in the x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii)

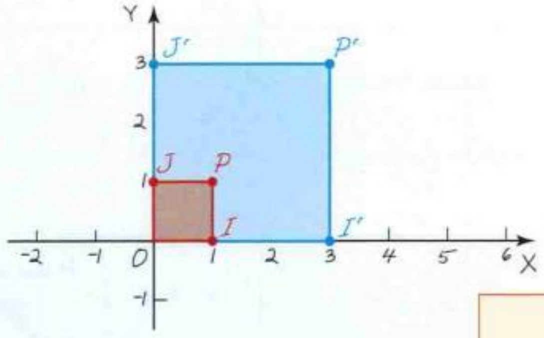


Figure 1.18

So the image of $\mathbf{I}$ is $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

You can see from Figure 1.18 that I (1, 0) is mapped to I′ (3, 0), and J (0, 1) is mapped to J′ (0, 3).

and the image of $\mathbf{J}$ is $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

So the matrix which represents an enlargement, centre the origin, scale factor 3 is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Discussion point

➜ For a general transformation represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, what are the images of the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

➜ What is the image of the origin (0, 0)?

ACTIVITY 1.3

Using the image of the unit square, find the matrix which represents a rotation of 45° anticlockwise about the origin.

Use your answer to write down the matrices which represent the following transformations:

(i) a rotation of 45° clockwise about the origin

(ii) a rotation of 135° anticlockwise about the origin.

Example 1.6

(i) Find the matrix which represents a rotation through angle θ anticlockwise about the origin.

(ii) Use your answer to find the matrix which represents a rotation of 60° anticlockwise about the origin.

Solution

(i) Figure 1.19 shows a rotation of angle θ anticlockwise about the origin.

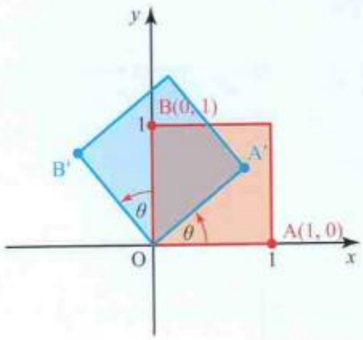


Figure 1.19

Call the coordinates of the point A′ (p, q). Since the lines OA and OB are perpendicular, the coordinates of B′ will be $(-q, p)$.

From the right-angled triangle with OA′ as the hypotenuse, $\cos\theta = \frac{p}{1}$ and so $p = \cos\theta$.

Similarly $\sin\theta = \frac{q}{1}$ so $q = \sin\theta$.

So, the image point A′ (p, q) has position vector $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and the image point B′ $(-q, p)$ has position vector $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

Therefore, the matrix that represents a rotation of angle θ anticlockwise about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(ii) The matrix that represents an anticlockwise rotation of 60° about the origin is $\begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

Discussion point

→ What matrix would represent a rotation through angle θ clockwise about the origin?

TECHNOLOGY

You could use geometrical software to try different values of m and n.

ACTIVITY 1.4

Investigate the effect of the matrices:

(i) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

Describe the general transformation represented by the matrices $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$.

Activity 1.4 illustrates two important general results.

- The matrix $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ represents a stretch of scale factor m parallel to the x-axis.
- The matrix $\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$ represents a stretch of scale factor n parallel to the y-axis.

Summary of transformations in two dimensions

Note

All these transformations are examples of linear transformations. In a linear transformation, straight lines are mapped to straight lines, and the origin is mapped to itself.

Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Reflection in the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Rotation anticlockwise about the origin through angle θ	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	Enlargement, centre the origin, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Stretch parallel to the x-axis, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	Stretch parallel to the y-axis, scale factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Transformations in three dimensions

When working with matrices, it is sometimes necessary to refer to a 'plane' – this is an infinite two-dimensional flat surface with no thickness. Figure 1.20 illustrates some common planes in three dimensions – the XY plane, the XZ plane and YZ plane. These three planes will be referred to when using matrices to represent some transformations in three dimensions. The plane XY can also be referred to as $z = 0$, since the z-coordinate would be zero for all points in the XY plane. Similarly, the XZ plane is referred to as $y = 0$ and the YZ plane as $x = 0$.

Answers

Chapter 1

Discussion point (Page 1)

3, 2, 1, 0

Discussion point (Page 4)

When subtracting numbers, the order in which the numbers appear is important – changing the order changes the answer, for example: $3 - 6 \neq 6 - 3$. So subtraction of numbers is not commutative.

The grouping of the numbers is also important, for example $(13 - 5) - 2 \neq 13 - (5 - 2)$. Therefore subtraction of numbers is not associative.

Matrices follow the same rules for commutativity and associativity under addition and subtraction as numbers. Matrix addition is both commutative and associative, but matrix subtraction is not commutative or associative. This is true because addition and subtraction of each of the individual elements will determine whether the matrices are commutative or associative overall.

You can use more formal methods to prove these properties. For example, to show that matrix addition is commutative:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Addition of numbers is commutative

Exercise 1.1 (Page 4)

1 (i) 3×2 (ii) 3×3 (iii) 1×2
(iv) 5×1 (v) 2×4 (vi) 3×2

2 (i) $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 2 & 12 \end{pmatrix}$

(iii) $\begin{pmatrix} -8 & 5 \\ -3 & 7 \end{pmatrix}$

(iv) Non-conformable (v) $\begin{pmatrix} -3 & -9 & 14 \\ 0 & 0 & 4 \end{pmatrix}$

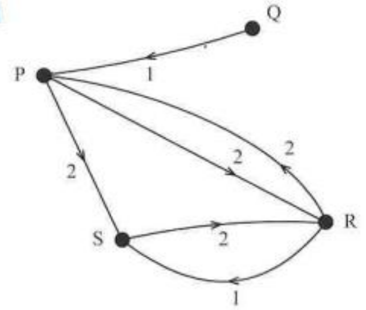
(vi) $\begin{pmatrix} 4 \\ 12 \\ 20 \end{pmatrix}$ (vii) $\begin{pmatrix} 9 & 7 & -17 \\ 10 & 5 & 28 \end{pmatrix}$

(viii) Non-conformable

(ix) $\begin{pmatrix} -15 & 8 \\ -4 & 3 \end{pmatrix}$

3 (i) $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$

(iii)



4 $w = 2, x = -6, y = -2, z = 2$

5 $p = -1$ or 6, $q = \pm\sqrt{5}$

6 (i) $\begin{pmatrix} 1 & 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 7 & 5 \\ 0 & 1 & 0 & 3 & 3 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 & 1 & 10 & 7 \\ 0 & 0 & 4 & 2 & 10 \\ 3 & 1 & 1 & 11 & 8 \\ 1 & 2 & 1 & 8 & 6 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$

City 2 vs United 1
Rangers 2 vs Town 1
Rangers 1 vs /United 1

7 (i) $\begin{pmatrix} 15 & 3 & 7 & 15 \\ 5 & 9 & 15 & -3 \\ 19 & 10 & 9 & 3 \end{pmatrix}$

The matrix represents the number of jackets left in stock after all the orders have been dispatched. The negative element indicates there was not enough of that type of jacket in stock to fulfil the order.

(ii) $\begin{pmatrix} 20 & 13 & 17 & 20 \\ 15 & 19 & 20 & 12 \\ 19 & 10 & 14 & 8 \end{pmatrix}$

(iii) $\begin{pmatrix} 12 & 30 & 18 & 0 \\ 6 & 18 & 24 & 36 \\ 30 & 0 & 12 & 18 \end{pmatrix}$

Probably not very realistic, as a week is quite a short time.

Discussion point (Page 8)

The dimensions of the matrices are **A** (3 × 3), **B** (3 × 2) and **C** (2 × 2). The conformable products are **AB** and **BC**. Both of these products would have dimension (3 × 2), even though the original matrices are not the same sizes.

Activity 1.1 (Page 9)

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -6 & -1 \\ -20 & 4 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -8 & 4 \\ -1 & 6 \end{pmatrix}$$

These two matrices are not equal and so matrix multiplication is not usually commutative. There are some exceptions, for example if

$\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ then

$$\mathbf{CD} = \mathbf{DC} = \begin{pmatrix} 6 & 6 \\ -2 & -2 \end{pmatrix}.$$

Activity 1.2 (Page 10)

(i) $\mathbf{AB} = \begin{pmatrix} -6 & -1 \\ -20 & 4 \end{pmatrix}$

(ii) $\mathbf{BC} = \begin{pmatrix} -4 & -8 \\ 0 & -1 \end{pmatrix}$

(iii) $(\mathbf{AB})\mathbf{C} = \begin{pmatrix} -8 & -15 \\ -12 & -28 \end{pmatrix}$

(iv) $\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} -8 & -15 \\ -12 & -28 \end{pmatrix}$

$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ so matrix multiplication is associative in this case

To produce a general proof, use general matrices such as

$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$.

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

$$\mathbf{BC} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} ei + fk & ej + fl \\ gi + hk & gj + hl \end{pmatrix}$$

and so

$$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

$$\begin{pmatrix} aei + bgi + afk + bhk & aej + bgj + afl + bhl \\ cei + cfk + dgi + dhk & cej + cfl + dgj + dhl \end{pmatrix}$$

and

$$\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} ei + fk & ej + fl \\ gi + hk & gj + hl \end{pmatrix}$$

$$= \begin{pmatrix} aei + afk + bgi + bhk & aej + afl + bgj + bh \\ cei + dgi + cfk + dhk & cej + dgj + cfl + dh \end{pmatrix}$$

Since $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ matrix multiplication is associative and the product can be written without brackets as **ABC**.

Exercise 1.2 (Page 10)

1 (i) (a) 3 × 3 (b) 1 × 3 (c) 2 × 3 (d) 2 × 4 (e) 2 × 1 (f) 3 × 5

(ii) (a) non-conformable

(b) 3 × 5

(c) non-conformable

(d) 2 × 3

(e) non-conformable

2 (i) $\begin{pmatrix} 21 & 6 \\ 31 & 13 \end{pmatrix}$ (ii) $\begin{pmatrix} -30 & -15 \end{pmatrix}$

(iii) $\begin{pmatrix} -54 \\ -1 \end{pmatrix}$

3 $\mathbf{AB} = \begin{pmatrix} 3 & -56 \\ 20 & -73 \end{pmatrix}$, $\mathbf{BA} = \begin{pmatrix} -25 & 8 \\ 28 & -45 \end{pmatrix}$

$\mathbf{AB} \neq \mathbf{BA}$ so matrix multiplication is non-commutative.

4 (i) $\begin{pmatrix} -7 & 26 \\ 2 & 34 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & 25 \\ 16 & 22 \end{pmatrix}$

(iii) $\begin{pmatrix} 31 & 0 \\ 65 & 18 \end{pmatrix}$ (iv) $\begin{pmatrix} 26 & 37 & 16 \\ 14 & 21 & 28 \\ -8 & -11 & 2 \end{pmatrix}$

(v) non-conformable (vi) $\begin{pmatrix} 28 & -18 \\ 26 & 2 \\ 16 & 25 \end{pmatrix}$

5 $\begin{pmatrix} -38 & -136 & -135 \\ 133 & 133 & 100 \\ 273 & 404 & 369 \end{pmatrix}$

6 (i) $\begin{pmatrix} 2x^2 + 12 & -9 \\ -4 & 3 \end{pmatrix}$ (ii) $x = 2$ or 3

(iii) $\mathbf{BA} = \begin{pmatrix} 8 & 12 \\ 8 & 15 \end{pmatrix}$ or $\begin{pmatrix} 18 & 18 \\ 12 & 15 \end{pmatrix}$

7 (i) (a) $\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 7 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 16 & 15 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$

8 (i) $\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 4 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 2 & 1 & 5 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$ $\mathbf{M}^2$ represents the number of two-stage routes between each pair of resorts.

(iii) $\mathbf{M}^3$ would represent the number of three-stage routes between each pair of resorts.

9 (i) $\begin{pmatrix} 8 + 4x & -20 + x^2 \\ x - 8 & -3 - 3x \end{pmatrix}$

(ii) $x = -3$ or 4

(iii) $\begin{pmatrix} -4 & -11 \\ -11 & 6 \end{pmatrix}$ or $\begin{pmatrix} 24 & -4 \\ -4 & -15 \end{pmatrix}$

10 (i) $\mathbf{D} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

$\mathbf{DA} = \begin{pmatrix} 299 & 199 & 270 & 175 & 114 \end{pmatrix}$

(ii) $\mathbf{F} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{AF} = \begin{pmatrix} 229 \\ 231 \\ 263 \\ 334 \end{pmatrix}$

(iii) $\mathbf{S} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{DAS} = (413)$,

$\mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{DAC} = (644)$

(iv) $\mathbf{P} = \begin{pmatrix} 0.95 \\ 0.95 \\ 1.05 \\ 1.15 \\ 1.15 \end{pmatrix}$,

$\mathbf{DAP} = (1088.95) = £1088.95$

11 (i) $\begin{pmatrix} b \\ a \\ c \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} b \\ c \\ a \end{pmatrix}$

(iv) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} c \\ a \\ b \end{pmatrix}$

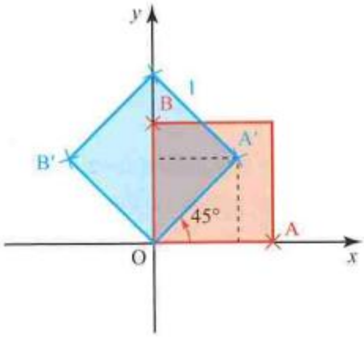
(v) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ The strands are back in the original order at the end of Stage 6.

Discussion point (Page 17)

The image of the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} a \\ c \end{pmatrix}$ and the image of the unit vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} b \\ d \end{pmatrix}$.

Activity 1.3 (Page 17)

The diagram below shows the unit square with two of its sides along the unit vectors $\mathbf{i}$ and $\mathbf{j}$. It is rotated by 45° about the origin.



You can use trigonometry to find the images of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

For A′, the x-coordinate satisfies $\cos 45 = \frac{x}{1}$ so $x = \cos 45 = \frac{1}{\sqrt{2}}$.

In a similar way, the y-coordinate of A′ is $\frac{1}{\sqrt{2}}$.

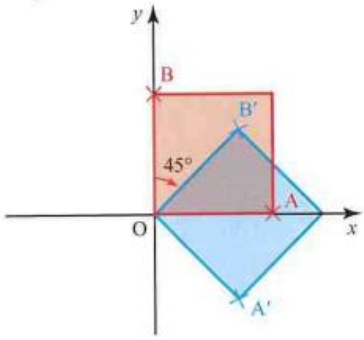
For B′, the symmetry of the diagram shows that the x-coordinate is $-\frac{1}{\sqrt{2}}$ and the y-coordinate is $\frac{1}{\sqrt{2}}$.

Hence, the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and so the matrix representing an anticlockwise rotation of 45° about the origin is $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

Rotations of 45° clockwise about the origin and 135° anticlockwise about the origin are also represented by matrices involving $\pm\frac{1}{\sqrt{2}}$.

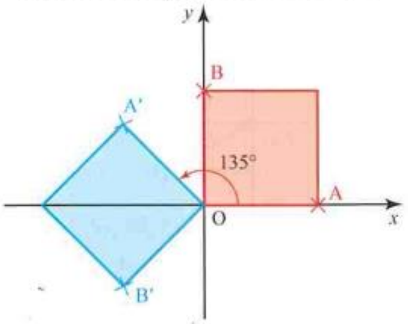
This is due to the symmetry about the origin.

(i) The diagram for a 45° clockwise rotation about the origin is shown below.



The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ and the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and so the matrix representing an anticlockwise rotation of 45° about the origin is $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

(ii) The diagram for a 135° anticlockwise rotation about the origin is shown below.



The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ and so the matrix representing

an anticlockwise rotation of 45° about the origin is

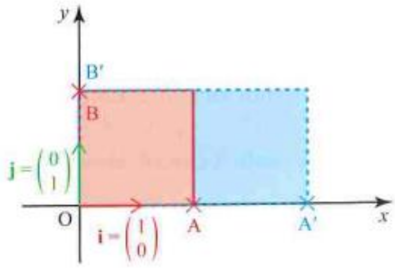
$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Discussion point (Page 18)

The matrix for a rotation of $\theta°$ clockwise about the origin is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

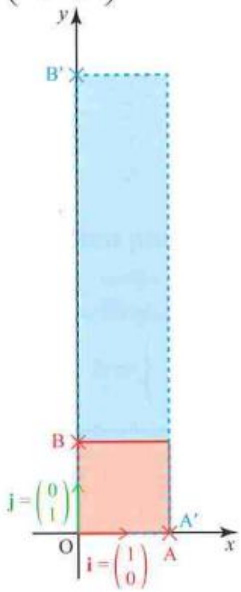
Activity 1.4 (Page 19)

(i) The diagram below shows the effect of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ on the unit vectors **i** and **j**.



You can see that the vector **i** has image $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and the vector **j** is unchanged. Therefore this matrix represents a stretch of scale factor 2 parallel to the x-axis.

(ii) The diagram below shows the effect of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ on the unit vectors **i** and **j**.



You can see that the vector **i** is unchanged and the vector **j** has image $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$. Therefore this matrix represents a stretch of scale factor 5 parallel to the y-axis.

The matrix $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ represents a stretch of scale factor m

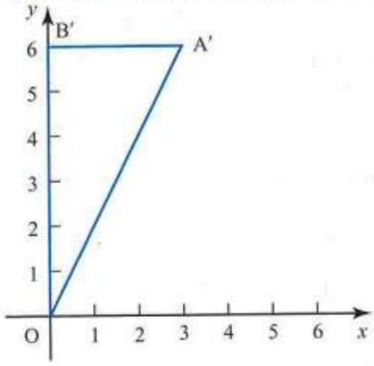
parallel to the x-axis.

The matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ represents a stretch of scale factor n

parallel to the y-axis.

Exercise 1.3 (Page 21)

1 (i) (a)

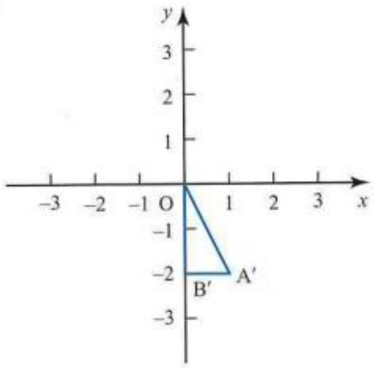


(b) $A' = (3, 6), B' = (0, 6)$

(c) $x' = 3x, y' = 3y$

(d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

(ii) (a)

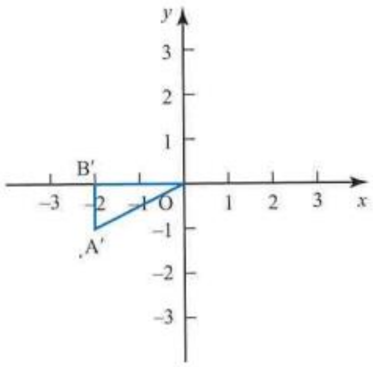


(b) $A' = (1, -2), B' = (0, -2)$

(c) $x' = x, y' = -y$

(d) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) (a)

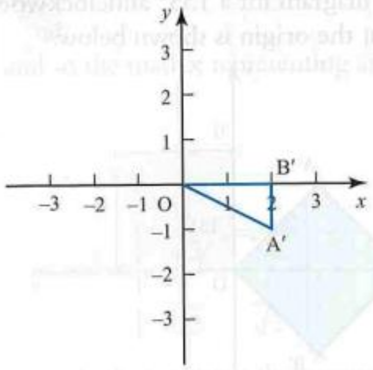


(b) $A' = (-2, -1), B' = (-2, 0)$

(c) $x' - y\ y' = -x$

(d) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(iv) (a)

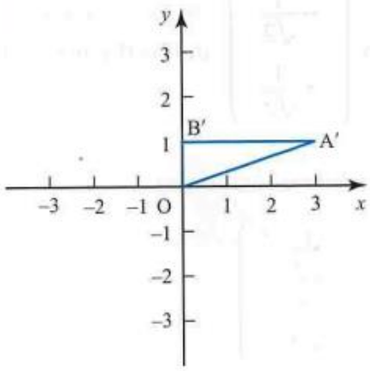


(b) $A' = (2, -1), B' = (2, 0)$

(c) $x' = y, y' = -x$

(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(v) (a)



(b) $A' = (3, 1), B' = (0, 1)$

(c) $x' = 3x, y' = \frac{1}{2}y$

(d) $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

2 (i) Reflection in the x-axis

(ii) Reflection in the line $y = -x$

(iii) Stretch of factor 2 parallel to the x-axis and stretch factor 3 parallel to the y-axis

(iv) Enlargement, scale factor 4, centre the origin

(v) Rotation of 90° clockwise (or 270° anticlockwise) about the origin

3 (i) Rotation of 60° anticlockwise about the origin

(ii) Rotation of 55° anticlockwise about the origin

(iii) Rotation of 135° clockwise about the origin

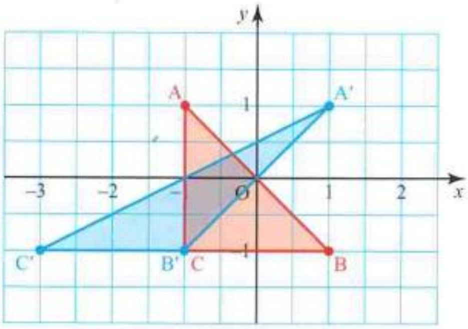
(iv) Rotation of 150° anticlockwise about the origin

4 (i) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

5 A′(4, 5), B′(7, 9), C′(3, 4). The original square and the image both have an area of one square unit.

6 (i)



(ii) The gradient of A′C′ is $\frac{1}{2}$, which is the reciprocal of the top right-hand entry of the matrix **M**.

7 (i) Rotation of 90° clockwise about the x-axis
(ii) Enlargement scale factor 3, centre (0, 0)
(iii) Reflection in the plane $z = 0$
(iv) Three-way stretch of factor 2 in the x-direction, factor 3 in the y-direction and factor 0.5 in the z-direction

8 (i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

9 $(x, y) \rightarrow (x, x)$

The matrix for the transformation is $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

10 (i) Any matrix of the form $\begin{pmatrix} 5 & 0 \\ 0 & k \end{pmatrix}$ or $\begin{pmatrix} k & 0 \\ 0 & 5 \end{pmatrix}$.

If $k = 5$ the rectangle would be a square.

(ii) $\begin{pmatrix} \sqrt{2} & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & \sqrt{2} \end{pmatrix}$,

$\begin{pmatrix} 1 & \sqrt{2} \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ \sqrt{2} & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 7 & \frac{3\sqrt{3}}{2} \\ 0 & \frac{3}{2} \end{pmatrix}$, $\begin{pmatrix} 0 & \frac{3}{2} \\ 7 & \frac{3\sqrt{3}}{2} \end{pmatrix}$,

$\begin{pmatrix} \frac{3\sqrt{3}}{2} & 7 \\ \frac{3}{2} & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \frac{3}{2} \\ 7 & \frac{3\sqrt{3}}{2} \end{pmatrix}$

Discussion point (Page 24)

(i) **BA** represents a reflection in the line $y = x$
(ii) The transformation A is represented by the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the transformation B is represented by the matrix

$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The matrix product

$\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

This is the matrix which represents a reflection in the line $y = x$.

Activity 1.5 (Page 24)

(i) $\mathbf{P}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

(ii) $\mathbf{P}'' = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

$= \begin{pmatrix} pax + pby + qcx + qdy \\ rax + rby + scx + sdy \end{pmatrix}$

(iii)

$$\mathbf{U} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} pa+qc & pb+qd \\ ra+sc & rb+sd \end{pmatrix}$$

and so

$$\mathbf{UP} = \begin{pmatrix} pa+qc & pb+qd \\ ra+sc & rb+sd \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} pax+qcx+pby+rdy \\ rax+scx+rby+sdy \end{pmatrix}.$$ Therefore $\mathbf{UP} = \mathbf{P}''$

Discussion point (Page 24)

AB represents 'carry out transformation B followed by transformation A.

(AB)C represents 'carry out transformation C followed by transformation AB, i.e. 'carry out C followed by B followed by A'.

BC represents 'carry out transformation C followed by transformation B'.

A(BC) represents 'carry out transformation BC followed by transformation A, i.e. carry out C followed by B followed by A'.

Activity 1.6 (Page 25)

(i) $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,

$\mathbf{B} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

(ii)

$$\mathbf{BA} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\sin\theta\cos\phi - \cos\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$$

(iii) $\mathbf{C} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$

(iv) $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

(v) A rotation through angle θ followed by rotation through angle ϕ has the same effect as a rotation through angle ϕ followed by angle θ.

Exercise 1.4 (Page 26)

1 (i) **A**: enlargement centre (0,0), scale factor 3
B: rotation 90° anticlockwise about (0,0)
C: reflection in the x-axis
D: reflection in the line $y = x$

(ii) $\mathbf{BC} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, reflection in the line $y = x$

$\mathbf{CB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, reflection in the line $y = -x$

$\mathbf{DC} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, rotation 90° anticlockwise about (0, 0)

$\mathbf{A}^2 = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$, enlargement centre (0, 0), scale factor 9

$\mathbf{BCB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, reflection in the x-axis

$\mathbf{DC^2D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ returns the object to its original position

(iii) For example, $\mathbf{B}^4$, $\mathbf{C}^2$ or $\mathbf{D}^2$

2 (i) $\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\mathbf{Y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $\mathbf{XY} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, rotation of 180° about the origin

(iii) $\mathbf{YX} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(iv) When considering the effect on the unit vectors **i** and **j**, as each transformation only affects one of the unit vectors the order of the transformations is not important in this case.

3 (i) $\mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) $\mathbf{PQ} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, reflection in the line $y = -x$

(iii) $\mathbf{QP} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(iv) The matrix **P** has the effect of making the coordinates of any point the negative of their original values,

i.e. $(x, y) \rightarrow (-x, -y)$

The matrix **Q** interchanges the coordinates,

i.e. $(x, y) \rightarrow (y, x)$

It does not matter what order these two transformations occur as the result will be the same

4 (i) $\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$\mathbf{L} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

(ii) (a) **LJ** (b) **MJ**
(c) $\mathbf{K}^2$ (d) **JLK**

5 (i) $\begin{pmatrix} 8 & -4 \\ -3 & 12 \end{pmatrix}$ (ii) $(32, -33)$

6 Possible transformations are $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is a rotation of 90° clockwise about the origin, followed by

$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, which is a stretch of scale factor 3 parallel to the x-axis. The order of these is important as performing **A** followed by **B** leads to the matrix $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$. Could also have

$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. which represents a stretch of factor 3 parallel to the y-axis, followed by

$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which represents a rotation of 90° clockwise about the origin; again the order is important.

7 $\mathbf{X} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

A matrix representing a rotation about the origin has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and so the entries on the leading diagonal would be equal. That is not true for matrix **X** and so this cannot represent a rotation.

8 $\mathbf{Y} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

9 (i) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(ii) A reflection in the x-axis and a stretch of scale factor 5 parallel to the x-axis

(iii) $\begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$

Reflection in the x-axis; stretch of scale factor 5 parallel to the x-axis; stretch of scale factor 2 parallel to the y-axis. The outcome of these three transformations would be the same regardless of the order in which they are applied. There are six different possible orders.

(iv) $\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

10 (i) $\begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & \frac{R_1}{R_2}+1 \end{pmatrix}$

(iv) $\begin{pmatrix} 1+\frac{R_1}{R_2} & -R_1 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$

The effect of Type B followed by Type A is different to that of Type A followed by Type B.

11 $a = \sqrt{\frac{\sqrt{2}+2}{4}}$ and $b = \sqrt{\frac{1}{2(\sqrt{2}+2)}}$

D represents an anticlockwise rotation of 22.5° about the origin.

By comparison to the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for an anticlockwise rotation of θ about the origin, a and b are the exact values of $\cos 22.5°$ and $\sin 22.5°$ respectively.

Discussion point (Page 28)

In a reflection, all points on the mirror line map to themselves.

In a rotation, only the centre of rotation maps to itself.

Exercise 1.5 (Page 31)

1 (i) Points of the form $(\lambda, -2\lambda)$

(ii) (0, 0)

(iii) Points of the form $(\lambda, -3\lambda)$

(iv) Points of the form $(2\lambda, 3\lambda)$

2 (i) x-axis, y-axis, lines of the form $y = mx$

(ii) x-axis, y-axis, lines of the form $y = mx$

(iii) no invariant lines

(iv) $y = x$, lines of the form $y = -x + c$

(v) $y = -x$, lines of the form $y = x + c$

3 (i) Any points on the line $y = \frac{1}{2}x$, for example (0, 0), (2, 1) and (3, 1.5)

(ii) $y = \frac{1}{2}x$

(iii) Any line of the form $y = -2x + c$

(iv) Using the method of Example 1.11 leads to the equations

$2m^2 + 3m - 2 = 0 \Rightarrow m = 0.5$ or -2

$(4 + 2m)c = 0 \Rightarrow m = -2$ or $c = 0$

If $m = 0.5$ then $c = 0$ so $y = \frac{1}{2}x$ is invariant.

If $m = -2$ then c can take any value and so $y = -2x + c$ is an invariant line.

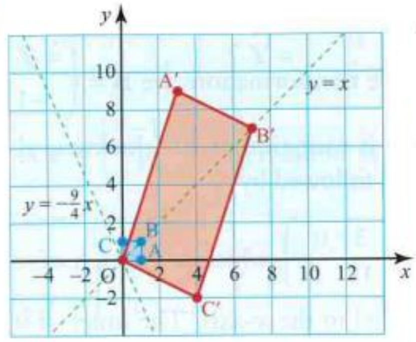
4 (i) Solving $\begin{pmatrix} 4 & 11 \\ 11 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ leads to the equations $y = -\frac{3x}{11}$ and $y = -\frac{11x}{3}$.

The only point that satisfies both of these is (0, 0).

(ii) $y = x$ and $y = -x$

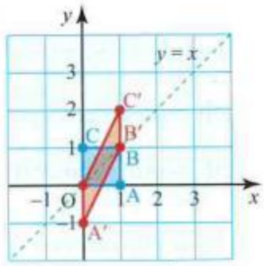
5 (i) $y = x$, $y = -\frac{9}{4}x$

(ii)



6 (i) $y = x$ (ii) $y = x$

(iii)



9 (i) $x' = x + a$, $y' = y + b$

(iii) (c) $a = -2b$